DISPERSIVE ESTIMATE FOR THE 1D SCHRÖDINGER EQUATION WITH A STEPLIKE POTENTIAL

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Abstract. We prove a sharp dispersive estimate

$$|P_{ac}u(t,x)| \le C|t|^{-1/2} \cdot \|u(0)\|_{L^1(\mathbb{R})}$$

for the one dimensional Schrödinger equation

$$iu_t - u_{xx} + V(x)u + V_0(x)u = 0,$$

where $(1+x^2)V \in L^1(\mathbb{R})$ and V_0 is a step function, real valued and consant on the positive and negative real axes.

1. Introduction

The *dispersive estimate* for the Schrödinger equation on $\mathbb{R}^n$, $n \ge 1$,

$$iu_t - \Delta u = 0, \qquad u(0,x) = f(x)$$

states that, for all $f \in L^1(\mathbb{R}^n)$ and $t \ne 0$, the solution satisfies

$$|u(t,x)| \le C|t|^{-\frac{n}{2}} \|f\|_{L^1(\mathbb{R}^n)}. \tag{1.1}$$

The sharp constant is $C = (4\pi)^{-n/2}$. Estimate (1.1) is elementary and follows from the explicit form of the fundamental solution; nevertheless, it represents the starting point for a large number of important developments including Strichartz estimates and the local and global well posedness theory for nonlinear Schrödinger equations. Thus the problem of extending (1.1) to more general equations has received a great deal of attention.

Potential perturbations of the form

$$iu_t - \Delta u + V(x)u = 0$$

(on $\mathbb{R}^n$, $n \ge 3$) were considered in many papers, starting with [18], with improvements at several reprises (see e.g. [25], [26], [27], [17], [11]). Focusing on the one dimensional case

$$iu_t - u_{xx} + V(x)u = 0, \tag{1.2}$$

which is the subject of this paper, the first proof of the dispersive estimate is surprisingly recent and due to Weder [22]. His result was improved by Artbazar and Yajima [2], who actually proved the more general fact that the wave operator associated to $-\frac{d^2}{dx^2} + V(x)$ is bounded on L^p for all p. Finally, Goldberg and Schlag [17] proved (1.1) for potentials satisfying $(1+x^2)V \in L^1(\mathbb{R})$, or the weaker condition $(1+|x|)V \in L^1(\mathbb{R})$ plus an additional nonresonant condition at 0; these conditions on the potential are conjectured to be optimal. Under the same assumptions on V, the L^p boundedness of the wave operator was proved by D'Ancona and Fanelli [10]. We also mention that potentials with slower decay present new phenomena as evidenced in [5], [6].

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Thus the problem of dispersive estimates for potential perturbations is essentially settled in 1D if the potential is small in the appropriate sense at infinity. Notice that we can add a real constant to V without modifying the dispersive properties via the gauge transformation $u \to ue^{i\lambda t}$. Hence a more precise statement is that dispersion has been proved for potentials having the same asymptotic behaviour at $x \to \pm\infty$.

Here we consider a more general kind of potential with possibly different asymptotic behaviours at $+\infty$ and $-\infty$. We call a *steplike potential* a potential of the form

$$V(x) + V_0(x)$$

where

$$V \in L^1(\mathbb{R}), \qquad V_0(x) = \begin{cases} V_- & : \quad x < 0 \\ V_+ & : \quad x > 0, \end{cases} \qquad V_-, V_+ \in \mathbb{R}.$$

It is not restrictive to assume $V_- < V_+$ as we shall do from now on.

In the physical literature, steplike potentials are also called *barrier potentials* and are used to model the interaction of particles with the boundary of solids (see [16] for a general discussion of problems with nontrivial asymptotics).

Steplike potentials occur also in general relativity. An example is given by the radial Klein-Gordon equation $\Box_g u - m^2 u = 0$ on a (radial) Schwarzschild background

$$g = -\left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2$$

Here $M > 0$ is related to the mass of the black hole and $r > 2M$ is a radial variable. If we introduce the Regge-Wheeler coordinate

$$s = r + 2M \log\left(\frac{r - 2M}{2M}\right) \tag{1.3}$$

and denote with $r(s)$ the inverse function of (1.3), the radial Klein-Gordon equation takes the form

$$u_{tt} - u_{ss} + v(s)u = 0, \qquad v(s) = \left(m^2 + \frac{2M}{r(s)^3}\right)\left(1 - \frac{2M}{r(s)}\right).$$

It is easy to check that the asymptotic behaviour of $v(s)$ is

$$v(s) \sim \begin{cases} m^2 + O(s^{-3}) & \text{for } s \to +\infty \\ O(e^{(2M)^{-1}s}) & \text{for } s \to -\infty, \end{cases}$$

so that the equation is reduced to a wave equation perturbed with a steplike potential.

Despite the significance of this class of potentials, mathematical studies have considered only the problem of direct and inverse scattering, with the usual applications to the Korteweg-de Vries equation, following the classical theory of [14] and [13]. A quite detailed theory was established in [4], [9], [12], [15], [1], [7], [8], [19]. See also [3] for a more recent take on this class of problems. .

On the other hand, to our knowledge, the dispersive properties of evolution equations perturbed with steplike potentials have never been investigated. Our goal here is to initiate this subject. The basic model is the Schrödinger equation

$$iu_t - u_{xx} + V_0(x)u = 0, \qquad u(0, x) = f(x) \tag{1.4}$$

where as above $V_0(x)$ is a piecewise constant function equal to V_+ for $x > 0$ and to V_- for $x < 0$, $V_- < V_+$. As a preparation to the study of more general potentials, in Section 2 we compute explicit kernels both for the resolvent of the operator

$-d^2/dx^2 + V_0$ and for the fundamental solution to (1.4). As a first application, we prove that any solution of (1.4) satisfies the dispersive estimate

$$|u(t,x)| \le C\|f\|_{L^1} \cdot |t|^{-1/2}, \qquad t \neq 0. \tag{1.5}$$

It is easy to check by a gauge transform and a rescaling $u \to e^{i\lambda t}u(\alpha^2 t, \alpha x)$ that the constant in the dispersive estimate does not depend on $V_\pm$.

After this preliminary study of the model case (1.5), we pass to the general situation of a Schrödinger equation

$$iu_t - u_{xx} + V_0 u + V(x)u = 0, \qquad u(0,x) = f(x) \tag{1.6}$$

where V_0 is perturbed with a potential $V(x)$ belonging to a suitable weighted L^1 class. In order to state our result, we recall that the operator $-d^2/dx^2 + V + V_0$ typically has a nonempty point spectrum, contained in $(-\infty, 0]$; since bound states do not disperse we need to project them away. We denote by P_{ac} the projection on the absolutely continuous subspace of $L^2(\mathbb{R})$ associated to the operator. Then the main result of the paper is the following:

Theorem 1.1. *For any $f \in L^1(\mathbb{R})$ and any real valued potential $V(x)$ satisfying*

$$(1 + x^2)V(x) \in L^1(\mathbb{R}), \tag{1.7}$$

the solution $u(t,x)$ to (1.6) satisfies the dispersive estimate

$$|P_{ac}u(t,x)| \le C\|f\|_{L^1} \cdot |t|^{-1/2}. \tag{1.8}$$

Section 3 is devoted to the proof of Theorem 1.1. As usual, we need to treat high frequencies and low frequencies with separate methods. Notice that for the high frequency part of the solution, dispersion can be proved under the weaker assumption $V \in L^1(\mathbb{R})$. The low frequency part is more difficult to estimate, and requires some rather precise information on the asymptotic behaviour of Jost solutions for the corresponding Helmholtz equation.

Remark 1.1. It is natural to question the optimality of the assumption $(1+x^2)V \in L^1$. A reasonable conjecture is that $(1 + x^2)^{\gamma/2}V(x) \in L^1$ with $\gamma \ge 1$ should be enough, provided some spectral assumption is made to exclude resonance at 0. In the classical case $V_0 \equiv 0$, the standard assumption is that the equation

$$-f'' + V(x)f = 0$$

has two linearly independent solutions f_+, f_- with the asymptotic behaviour

$$f_+ \sim 1 \text{ as } x \to +\infty, \qquad f_- \sim 1 \text{ as } x \to -\infty.$$

Potentials $V(x)$ satisfying this condition are called *generic*, while they are called *exceptional* when the condition fails. Notice that $V \equiv 0$ is of exceptional type. When V_0 is not zero, e.g., the Heaviside function, the asymptotic for f_+ should be modified to require $f_+ \sim e^{-x}$ as $x \to +\infty$. We prefer not to pursue here the delicate question of the optimal assumptions on the potential.

The main application of dispersive estimates concerns problems of global well posedness and stability for nonlinear perturbations of the equation. We shall devote further works to this aspect, while here we shall focus on the linear estimates exclusively.

2. Resolvent operator and dispersive estimate for $-d^2/dx^2 + V_0$

Throughout the paper, for complex numbers z we denote by

$$z^{1/2} \text{ the square root of } z \text{ with } \Im z^{1/2} \ge 0.$$

For any $z \notin \mathbb{R}$, let $R_0(z) = (H_0 - z)^{-1}$ be the resolvent of the selfadjoint operator $H_0 = -\frac{d^2}{dx^2} + V_0$, where V_0 is the piecewise constant function

$$V_0 = V_- \quad \text{for } x < 0, \qquad V_0 = V_+ \quad \text{for } x > 0.$$

Both the selfadjointness of H_0 and the fact that $\sigma(H_0) = [V_-, +\infty)$ follow from the standard theory. Denoting by $r_\pm(z)$ the functions

$$r_\pm = (z - V_\pm)^{1/2}, \qquad z \notin [V_-, +\infty) \tag{2.1}$$

we see that R_0 can be represented as an integral operator

$$R_0(z)f = \int K_z^0(x,y)f(y)dy$$

where the kernel $K_z^0(x,y)$ is expressed by the following formulas: for all $y < x \in \mathbb{R}$,

$$K_z^0(x,y) = K_z^0(y,x) = \begin{cases} \frac{1}{2ir_-}e^{ir_-(x-y)} + \frac{1}{2ir_-}\frac{r_- - r_+}{r_- + r_+}e^{ir_-(-x-y)} & \text{if } y < x < 0, \\ \frac{1}{i(r_+ + r_-)}e^{ir_+x}e^{-ir_-y} & \text{if } y < 0 < x, \\ \frac{1}{2ir_+}\frac{r_- - r_+}{r_- + r_+}e^{ir_+(x+y)} + \frac{1}{2ir_+}e^{ir_+(x-y)} & \text{if } 0 < y < x. \end{cases} \tag{2.2}$$

The explicit formula for K_z^0 can be computed by an elementary application of the standard theory of ordinary differential equations. Notice that the kernel has a well defined limit as z approaches a point of the spectrum of H_0 from above and from below (with different limits).

The spectral formula allows to represent any function $\phi(H_0)$ of the operator, for sufficiently nice ϕ, as the L^2 limit

$$\phi(H_0)f = \lim_{\epsilon \downarrow 0} \int_{\gamma_\epsilon} \phi(z)R_0(z)f dz \tag{2.3}$$

over a curve γ_ϵ which in the present case can be taken as the union of the straight half lines $z = \lambda \pm i\epsilon$, $\lambda > V_-$, with the left semicircle of radius ϵ around $z = V_-$, from $+\infty - i\epsilon$ to $+\infty + i\epsilon$. After the change of variables $z = w^2 + V_-$ we can rephrase the spectral formula as

$$\phi(H_0)f = \int_{-\infty}^{+\infty} \phi(\lambda + V_-)\lambda R_0(\lambda^2 + V_-)f d\lambda \tag{2.4}$$

where the limit operator

$$R_0(\lambda^2 + V_-)f = \int K_\lambda(x,y)f(y)dy$$

has a kernel K_λ given by the following expressions valid for all $y < x \in \mathbb{R}$:

$$K_\lambda(x,y) = K_\lambda(y,x) = \begin{cases} \frac{1}{2i\lambda}e^{i\lambda(x-y)} + \frac{1}{2i\lambda}\frac{\lambda - \rho_+}{\lambda + \rho_+}e^{i\lambda(-x-y)} & \text{if } y < x < 0, \\ \frac{1}{i(\rho_+ + \lambda)}e^{i\rho_+x}e^{-i\lambda y} & \text{if } y < 0 < x, \\ \frac{1}{2i\rho_+}\frac{\lambda - \rho_+}{\lambda + \rho_+}e^{i\rho_+(x+y)} + \frac{1}{2i\rho_+}e^{i\rho_+(x-y)} & \text{if } 0 < y < x \end{cases} \tag{2.5}$$

and the function $\rho_+(\lambda)$ is defined as

$$\rho_+(\lambda) = \begin{cases} (\lambda^2 - \delta^2)^{1/2} & \text{if } \lambda > \delta, \\ i(\delta^2 - \lambda^2)^{1/2} & \text{if } -\delta < \lambda < \delta, \\ -(\lambda^2 - \delta^2)^{1/2} & \text{if } \lambda < -\delta. \end{cases} \qquad \delta^2 = V_+ - V_- \geq 0. \tag{2.6}$$

Notice that (2.5) is a simple instance of the limiting absorption principle; indeed, the kernel K_λ is a bounded function and hence defines a bounded operator from $L^2(\langle x \rangle^{1+}dx)$ to $L^2(\langle x \rangle^{-1-}dx)$ as in the standard theory.

We can now use (2.4), (2.5), (2.6) to represent the solution of the Schrödinger equation (1.4) via the formula

$$e^{itH_0}f = e^{itV_-}\int_{-\infty}^{+\infty}\left(\int_{\mathbb{R}} e^{it\lambda^2}\lambda K_\lambda(x,y)f(y)dy\right)d\lambda. \tag{2.7}$$

The factor e^{itV_-} is inessential and will be dropped from now on (i.e. we can assume $V_- = 0$). Moreover, as remarked in the Introduction, by a rescaling we can assume $\delta = 1$. Thus from now on the function V_0 will be taken equal to the Heaviside function

$$V_0(x) = \mathbf{1}_+(x) = \mathbf{1}_{[0,+\infty)}(x).$$

In order to swap the integrals and perform some stationary phase calculations we introduce a cutoff function with the following properties: $\chi \in C_c^\infty(\mathbb{R})$ is an even function,

$$\chi(\lambda) = \begin{cases} 0 & \text{if } |\lambda| \geq 8, \\ 1 & \text{if } |\lambda| \leq 4, \end{cases} \qquad |\chi'| \leq 1, \qquad |\chi''| \leq 1. \tag{2.8}$$

Then we shall study the operator

(2.9)
$$\chi_M(H_0)e^{itH_0}f = \int_{\mathbb{R}}\int_{-\infty}^{+\infty} e^{it\lambda^2}\lambda\chi_M(\lambda)K_\lambda(x,y)d\lambda\ f(y)dy, \qquad \chi_M(\lambda) = \chi\left(\frac{\lambda}{M}\right).$$

Notice that, in order to prove (1.5), it will be sufficient to prove a dispersive $L^1 - L^\infty$ estimate for the truncated operator (2.9), uniform in $M >> 1$.

Finally, it is clearly sufficient to prove the pointwise estimate

$$\left|\int_{-\infty}^{+\infty} e^{it\lambda^2}\lambda\chi_M(\lambda)K_\lambda(x,y)d\lambda\right| \leq C|t|^{-1/2} \tag{2.10}$$

for some constant C independent of M, x and y. The rest of this section is devoted to the proof of (2.10). By symmetry we can assume $y < x$, and recalling (2.5) we shall handle separately the three cases $y < x < 0$, $y < 0 < x$ and $0 < y < x$.

2.1. **First case:** $y < x < 0$**.** Apart from inessential numeric factors, we must estimate the two terms

$$v(t,A) = \int \chi_M(\lambda)e^{it\lambda^2}e^{i\lambda A}d\lambda, \qquad A = x - y > 0 \tag{2.11}$$

and

$$w(t,A) = \int \chi_M(\lambda)\frac{\lambda-\rho_+}{\lambda+\rho_+}e^{it\lambda^2}e^{i\lambda A}d\lambda, \qquad A = -x - y > 0 \tag{2.12}$$

uniformly in M and $y < x < 0$. By a standard trick we regard $v(t,A)$ as a solution of the 1D Schrödinger equation

$$iv_t - v_{AA} = 0$$

and hence we can apply the usual dispersive estimate

$$|v| \leq |t|^{-1/2}\|v(0,A)\|_{L^1_A}, \qquad \|v(0,A)\|_{L^1_A} = \|\widehat{\chi_M}\|_{L^1} = \|\widehat{\chi}\|_{L^1}.$$

Thus we get

$$|v| \leq \|\widehat{\chi}\|_{L^1}\cdot|t|^{-1/2} \tag{2.13}$$

A similar strategy applied to $w(t,A)$ gives

$$|w| \leq |t|^{-1/2}\|I\|_{L^1_A}$$

with

$$I = \int \frac{\lambda-\rho_+}{\lambda+\rho_+}e^{i\lambda A}\chi_M(\lambda)d\lambda. \tag{2.14}$$

Notice that for $\lambda > 1$ (recall that $\delta = 1$)

$$\frac{\lambda - \rho_+}{\lambda + \rho_+} = \frac{1}{(\lambda + \rho_+)^2} = \frac{1}{\left(\lambda + (\lambda - 1)^{1/2}\right)^2}$$

and analogously for $\lambda < -1$, $-1 < \lambda < 1$, thus we have

$$I = \int \frac{1}{(\lambda + \rho_+)^2} e^{i\lambda A} \chi_M d\lambda.$$

Introduce now an additional cutoff $\psi(\lambda)$ in the integral to isolate the Hölder singularity around $\lambda = 1$ of ρ_+, i.e., choose $\psi \in C^\infty(\mathbb{R})$ with

$$\psi(\lambda) = \begin{cases} 1 & \text{if } \lambda \geq 3, \\ 0 & \text{if } \lambda \leq 2 \end{cases}$$

and consider the corresponding piece of the integral

$$I_1 = \int \frac{\psi}{(\lambda + \rho_+)^2} e^{i\lambda A} \chi_M(\lambda) d\lambda. \tag{2.15}$$

Two integrations by parts give

$$\|I_1\|_{L^1_A} \leq \left\| \frac{\psi \chi_M}{(\lambda + \rho_+)^2} \right\|_{W^{2,1}_\lambda}$$

which is uniformly bounded for $M \geq 1$.

The piece of I near $\lambda = 1$ is more delicate. We choose a new cutoff which we denote again by $\psi \in C^\infty_c$ supported in $[-1/2, 3]$ and consider I_2 given by

$$I_2 = \int \frac{\lambda - \rho_+}{\lambda + \rho_+} e^{i\lambda A} \psi \chi_M d\lambda = \int \frac{\lambda - \rho_+}{\lambda + \rho_+} e^{i\lambda A} \psi d\lambda \tag{2.16}$$

where we can suppress χ_M since $\psi \chi_M \psi$ for $M > 1$. We decompose the singular factor into even and odd part using the identity

$$\frac{\lambda - s}{\lambda + s} = \frac{\lambda^2 + s^2}{\lambda^2 - s^2} - \frac{2s\lambda}{\lambda^2 - s^2}$$

obtaining (recall $\rho_+^2 = \lambda^2 - \delta^2 = \lambda^2 - 1$)

$$\frac{\lambda - \rho_+}{\lambda + \rho_+} = (2\lambda^2 - 1) - 2\lambda\rho_+.$$

The term in $(2\lambda^2 - 1)$ is trivial since (apart from numeric factors)

$$\int (2\lambda^2 - 1)\psi e^{i\lambda A} d\lambda = -2\widehat{\psi}''(A) - \widehat{\psi}(A)$$

which is certainly L^1 bounded. On the other hand, we can split

$$\int \lambda \psi \rho_+ e^{i\lambda A} d\lambda = \int_1^{+\infty} + \int_{-\infty}^1$$

and we shall focus on the first term since the second one is entirely analogous. We rewrite it as

$$\int_1^{+\infty} \lambda \psi \rho_+ e^{i\lambda A} d\lambda = \int_0^\infty \psi_1(\lambda) \lambda^{1/2} e^{i\lambda A} d\lambda \cdot e^{iA}, \qquad \psi_1(\lambda) = (\lambda + 1)^{3/2} \psi(\lambda + 1)$$

and notice that, again, $\psi_1 \in C^\infty_c$. This kind of integral is standard; applying e.g. estimate (A.2) in the Appendix we get

$$\left| \int_1^{+\infty} \lambda \psi \rho_+ e^{i\lambda A} d\lambda \right| \leq C(\psi) |A|^{-3/2},$$

while we have directly

$$\left| \int_1^{+\infty} \lambda \psi \rho_+ e^{i\lambda A} d\lambda \right| \leq C(\psi).$$

Summing up we obtain $\|I_2\|_{L^1_A} \leq C$ as required. The estimate of the remaining part of I is identical, using cutoffs supported in $(-\infty, -2]$ and $[-3.1/2]$ as above.

2.2. **Second case:** $y < 0 < x$**.** The quantity (2.10) takes the form $(y \to -y)$

$$\int \frac{\lambda}{\rho_+ + \lambda} e^{i\rho_+ x} e^{i\lambda y} e^{i\lambda^2 t} \chi_M d\lambda, \qquad x, y > 0.$$

We split the integral in three parts

$$I + II + III = \int_1^{+\infty} + \int_{-1}^{1} + \int_{-\infty}^{-1}.$$

After a reflection $\lambda \to -\lambda$ and a conjugation in III we see that parts I and III have the form

$$I, III = \int_1^{\infty} \frac{\lambda}{\rho_+ + \lambda} e^{i\rho_+ x} e^{i\lambda y} e^{\pm i\lambda^2 t} \chi_M d\lambda, \qquad x, y > 0.$$

The basic tool will be the following Lemma:

Lemma 2.1. *For all $b > 1$, $x, y > 0$,*

$$\left| \int_1^b \frac{\lambda}{\rho_+ + \lambda} e^{i\rho_+ x} e^{i\lambda y} e^{\pm i\lambda^2 t} \right| \leq 15|t|^{-1/2}. \tag{2.17}$$

Proof. We recall one form of the van der Corput Lemma (see [20]: for any C^2 function $\phi : [a, b] \to \mathbb{R}$ such that $|\phi''| \geq 1$ on $[a, b]$,

$$\left| \int_a^b e^{i\phi(\lambda)t} \psi(\lambda) d\lambda \right| \leq 10|t|^{-1/2} \left[\psi(b) + \int_a^b |\psi'(\lambda)| d\lambda \right] \tag{2.18}$$

Consider the $+$ case in (2.17); we can assume $t > 0$. After the change of variables

$$\mu = \rho_+(\lambda) \implies \lambda = \langle \mu \rangle, \quad \lambda \, d\lambda = \mu d\mu$$

the integral becomes

$$I_+ = \int_0^{\sqrt{b^2-1}} \frac{\mu}{\mu + \langle \mu \rangle} e^{i\mu x} e^{i\langle \mu \rangle y} e^{i(\mu^2+1)t} d\mu.$$

We apply now (2.18) with the choice

$$\phi(\mu) = \mu^2 + 1 + \mu \frac{x}{t} + \langle \mu \rangle \frac{y}{t} \qquad \implies \qquad \phi'' = 2 + \frac{1}{\langle \mu \rangle^3} \frac{y}{t} \geq 2$$

and we obtain

$$|I_+| \leq \frac{10}{|t|^{1/2}} \left[\frac{\sqrt{b^2-1}}{b + \sqrt{b^2-1}} + \int_0^{\sqrt{b^2-1}} \left| \left(\frac{\mu}{\mu + \langle \mu \rangle} \right)' \right| d\mu \right].$$

Noticing that

$$\left(\frac{\mu}{\mu + \langle \mu \rangle} \right)' = \frac{1}{\langle \mu \rangle (\mu + \langle \mu \rangle)^2}$$

we arrive at (2.17).

Consider now the case of a minus sign in (2.17). We take as a phase function

$$\phi = -\lambda^2 + \frac{y}{t}\lambda + \frac{x}{t}\sqrt{\lambda^2 - 1} \qquad \implies \qquad \phi'' = -2 - \frac{1}{(\lambda^2-1)^{3/2}} \frac{x}{t} \leq -2$$

(the singularity of the phase at $\lambda = 1$ can obviously be overcome by restricting on an interval $[1+\epsilon, b]$ and letting $\epsilon \downarrow 0$, since the final estimate is uniform in ϵ). As above we obtain

$$|I_-| \leq \frac{10}{|t|^{1/2}} \left[\frac{b}{b+\sqrt{b^2-1}} + \int_1^b \left| \left(\frac{\lambda}{\lambda+\sqrt{\lambda^2-1}} \right)' \right| d\lambda \right].$$

Since

$$\left(\frac{\lambda}{\lambda+\sqrt{\lambda^2-1}} \right)' = \frac{1}{\sqrt{\lambda^2-1}(\lambda+\sqrt{\lambda^2-1})^2}$$

we conclude again with (2.17). $\square$

Now in order to estimate I (or III) it is sufficient to write it as

$$I = F *_x G$$

with

$$F = \int_0^{8M} \frac{\lambda}{\lambda+\rho_+} e^{i\rho_+ x} e^{i\lambda y} e^{i\lambda^2 t} d\lambda, \qquad G = \int e^{i\lambda x} \chi(\lambda/M) d\lambda$$

and apply Young's inequality and (2.17)

$$|I| \leq \|F\|_{L^\infty_{x,y}} \|G\|_{L^1_x} \leq 15|t|^{-1/2} \|\chi\|_{L^1}.$$

For the remaining piece II we need a refinement of the van der Corput Lemma (2.18) which is proved as follows. If the function $\psi(\lambda)$ is real valued and monotonic on $[a, b]$, we can rewrite (2.18) in the equivalent form

$$\left| \int_a^b e^{i\phi(\lambda)t} \psi(\lambda) d\lambda \right| \leq 10|t|^{-1/2} \left[\psi(b) + \left| \int_a^b \psi'(\lambda) d\lambda \right| \right] \leq 30 \|\psi\|_{L^\infty} |t|^{-1/2}.$$

As a consequence, if the phase $\phi(\lambda)$ satisfies $|\phi''| \geq 1$ on (a, b) and if both $\Im\psi$ and $\Re\psi$ make at most N oscillations on the interval $[a, b]$, the following estimate holds

$$\left| \int_a^b e^{i\phi(\lambda)t} \psi(\lambda) d\lambda \right| \leq 60N \|\psi\|_{L^\infty} |t|^{-1/2} \tag{2.19}$$

independently of a, b.

Cnsider now the term II:

$$II = \int_{-1}^{+1} \frac{\lambda}{\rho_+ + \lambda} e^{i\rho_+ x} e^{i\lambda y} e^{it\lambda^2} d\lambda,$$

recalling that in the range $-1 < \lambda < 1$ we have $\rho_+(\lambda) = i(1-\lambda^2)^{1/2}$ and $\chi_M = 1$. We shall use estimate (2.19) with the choices

$$\phi = \lambda^2 + \lambda \frac{y}{t} \quad \Longrightarrow \quad \phi'' \geq 2$$

and

$$\psi = \frac{\lambda}{\rho_+ + \lambda} e^{i\rho_+ x} = (\lambda^2 - i\lambda(1-\lambda^2)^{1/2})) e^{-(1-\lambda^2)^{1/2} x}.$$

It is trivial to check that both the real and the imaginary part of ψ make at most 2 oscillations on $[-1, 1]$, while $|\psi| \leq 1$, and in conclusion

$$|II| \leq 120|t|^{-1/2}.$$

2.3. **Third case:** $0 < y < x$**.** Apart from a factor $2i$, the quantity (2.10) in this case is the sum of two terms

$$w(t,A) = \int \frac{\lambda}{\rho_+} e^{i\rho_+ A} e^{it\lambda^2} \chi_M d\lambda, \qquad A = x - y > 0$$

and

$$v(t,A) = \int \frac{\lambda}{\rho_+} \frac{\rho_+ - \lambda}{\rho_+ + \lambda} e^{i\rho_+ A} e^{it\lambda^2} \chi_M d\lambda, \qquad A = x + y > 0.$$

Now $w(t,A)$ solves the modified Schrödinger equation

$$iw_t - w_{AA} + \delta^2 w = 0 \qquad \Longrightarrow \qquad |w| = |e^{-i\delta^2 t} w| \le |t|^{-1/2} \|w(0,A)\|_{L^1_A},$$

and hence we are reduced to give a uniform L^1_A bound of the integral

$$I_0 = \int \frac{\lambda}{\rho_+} e^{i\rho_+ A} \chi_M d\lambda. \tag{2.20}$$

We notice that for $-1 < \lambda < 1$ the function $\rho_+(\lambda) = i(1-\lambda^2)^{1/2}$ is even and so is $\chi_M(\lambda)$, hence the part of the integral on $[-1,1]$ vanishes. Performing the change of variables

$$\mu = \rho_+(\lambda) \qquad \Longrightarrow \qquad \lambda = \operatorname{sgn}\mu \cdot \langle \mu \rangle, \qquad \frac{\lambda}{\rho_+} d\lambda = d\mu$$

in the remaining parts we get

$$I_0 = \int_{-\infty}^{-1} + \int_1^{+\infty} = \int_{-\infty}^{+\infty} e^{i\mu A} \chi_M(\langle \mu \rangle) d\lambda \tag{2.21}$$

since χ is an even function. Writing $\chi(\mu) = \widetilde{\chi}(\mu^2)$, $\widetilde{\chi} \in C_c^\infty$, we have then

$$\|I_0\|_{L^1_A} = \left\| \int e^{i\mu A} \widetilde{\chi}(\mu^2 + M^{-2}) d\mu \right\|_{L^1_A} \le \pi \int \left| (1 - \partial_\mu^2) \widetilde{\chi}(\mu^2 + M^{-2}) \right| d\mu$$

which is bounded independently of $M > 1$.

The second term $v(t,A)$ can be estimated directly using a stationary phase argument. For the region $\lambda > 1$ we change variable as above

$$\int_1^{+\infty} \frac{\lambda}{\rho_+} \frac{\rho_+ - \lambda}{\rho_+ + \lambda} e^{i\rho_+ A} e^{it\lambda^2} \chi_M d\lambda = e^{it} \int_0^{+\infty} \frac{\mu - \langle \mu \rangle}{\mu + \langle \mu \rangle} e^{i(\mu^2 + A/t)t} \chi_M(\langle \mu \rangle) d\mu$$

and we can apply the strong form of van der Corput (2.19) since the phase staisfies

$$\phi(\mu) = \mu^2 + \frac{A}{t} \qquad \Longrightarrow \qquad \phi'' \ge 2$$

while the amplitude

$$\frac{\mu - \langle \mu \rangle}{\mu + \langle \mu \rangle} \chi_M(\langle \mu \rangle) = (2\mu \langle \mu \rangle - 1 - 2\mu^2) \chi_M(\langle \mu \rangle)$$

is bounded by 1 and makes a finite number of oscillations on $(1, +\infty)$. Thus this part of $v(t,A)$ decays like $|t|^{-1/2}$, uniformly in M, A. An analogous argument gives the same bound for the region $\lambda < -1$. For remaining part of the integral on $-1 < \lambda < 1$, noticing that the cutoff is equal to 1 there, we split further into the piece

$$\int_0^{+1} \frac{\lambda}{\rho_+} \frac{\rho_+ - \lambda}{\rho_+ + \lambda} e^{i\rho_+ A} e^{it\lambda^2} d\lambda$$

and a symmetric one for $-1 < \lambda < 0$ which is estimated in an identical way. Changing variable as $\lambda = \sqrt{1 - \mu^2}$ the integral becomes

$$-ie^{it} \int_0^1 \frac{\mu - i\sqrt{1-\mu^2}}{\mu + i\sqrt{1-\mu^2}} e^{-A\sqrt{1-\mu^2}} e^{-it\mu^2} d\mu$$

and we can apply again (2.19) choosing as phase $\phi = -\mu^2$ and as amplitude

$$\frac{\mu - i\sqrt{1-\mu^2}}{\mu + i\sqrt{1-\mu^2}} e^{-A\sqrt{1-\mu^2}} = (2\mu^2 - 1 - 2i\mu\sqrt{1-\mu^2})e^{-A\sqrt{1-\mu^2}}.$$

Notice indeed that the amplitude is bounded since $A > 0$, and both its real and imaginary part make a finite number of oscillations on $(0, 1)$ independent of A. Thus also the last piece decays as $|t|^{-1/2}$ and the proof is concluded.

3. General step potentials

Consider now an L^1 perturbation of H_0 of the form ($\mathbf{1}_+ = \mathbf{1}_{[0,+\infty)}$)

$$H = -\frac{d^2}{dx^2} + \mathbf{1}_+(x) + V(x), \qquad V \in L^1(\mathbb{R}) \quad \text{real valued}, \qquad D(H) = H^2(\mathbb{R}).$$

By the standard theory (see e.g. [23], [24], [21]) H is a selfadjoint operator, and its spectrum decomposes as

$$\sigma(H) = \sigma_p(H) \cup \sigma_{ac}(H), \qquad \sigma_{ac}(H) = [0, +\infty), \qquad \sigma_p = \{\lambda_j\}_{j\geq 1} \subset (-\infty, 0].$$

The sequence of negative eigenvalues, if infinite, can accumulate only at 0.

3.1. **Low frequencies.** From now on we make the stronger assumption on V

$$(1 + x^2) \cdot V(x) \in L^1(\mathbb{R}) \tag{3.1}$$

(although for several results the weaker condition $(1 + |x|)V \in L^1$ would be sufficient). Then most of the standard theory of Jost solutions (see [13]) carries through to the case of step potentials, as proved in [9]. We recall the essential facts that we shall need in the following.

Consider the resolvent equation on $\mathbb{R}$

$$-\frac{d^2}{dx^2} f(z, x) + (\mathbf{1}_+ + V)f(z, x) = z^2 f(z, x), \qquad z \in \mathbb{C}, \quad \Im z \geq 0. \tag{3.2}$$

In order to describe the Jost solutions we extend the definition of the function $\rho_+(z)$ to the upper half plane $\{\Im z \geq 0\}$ as

$$\rho_+(z) = \text{ the branch of } (z^2 - 1)^{1/2} \text{ with nonnegative imaginary part.} \tag{3.3}$$

Notice that for real z this reduces precisely to (2.6). The function $\rho_+(z)$ is continuous on $\{\Im z \geq 0\}$ and analytic on $\{\Im z > 0\}$, and is a bijection of $\{\Im z > 0\}$ onto the upper half plane with a slit

$$\Omega = \{\Im z > 0\} \setminus S, \qquad S = \{z \in \mathbb{C} \colon \Re z = 0, \ 0 \leq \Im z \leq 1\}$$

with a jump across the slit S. Then, under assumption (3.1), for each z with $\Im z \geq 0$, equation (3.2) has two solutions $f_\pm(z, x)$ uniquely determined by the properties

$$e^{-i\rho_+(z)x} f_+(z, x) \to 1 \quad \text{and} \quad e^{-i\rho_+(z)x} f'_+(z, x) \to i\rho_+(z) \quad \text{as } x \to +\infty, \tag{3.4}$$

$$e^{izx} f_-(z, x) \to 1 \quad \text{and} \quad e^{izx} f'_-(z, x) \to -iz \quad \text{as } x \to -\infty. \tag{3.5}$$

Such solutions are called the *Jost solutions* of the resolvent equation (3.2). Their Wronskian

$$W(z) = W[f_+, f_-] = f_+(z, 0)\partial_x f_-(z, 0) - f_-(z, 0)\partial_x f_+(z, 0) \tag{3.6}$$

is continuous on $\{\Im z \geq 0\}$, analytic on $\{\Im z > 0\}$, and satisfies the fundamental property

$$W(\lambda) \neq 0 \quad \text{for } 0 \neq \lambda \in \mathbb{R}. \tag{3.7}$$

According to the standard terminology, when the Jost solutions are independent at $\lambda = 0$ i.e. when $W(0) \neq 0$, the potential $\mathbf{1}_+ + V$ is said to be of *generic type*, while in the case $W(0) = 0$ it is said to be of *exceptional type*.

Theorem 3.1 (Lemma 2.4 in [9]). *Assume the potential $V(x)$ satisfies*

$$\langle x\rangle^2 V(x) \in L^1(\mathbb{R}). \tag{3.8}$$

Then the Wronskian $W(z) = [f_+(z,x), f_-(z,x)]$ is continuous for $\Im z \geq 0$, analytic for $\Im z > 0$, and different from zero for all $z \in \mathbb{R}\setminus 0$. Moreover,

(i) either $W(0) \neq 0$,

(ii) or $W(0) = 0$ and for some real $\gamma \neq 0$

$$\dot{W}(0) = \lim_{\substack{z\to 0\\ \Im z\geq 0}} \frac{W(z)}{z} = i\gamma. \tag{3.9}$$

Now consider the resolvent equation $(H - z^2)u = h$ for $h \in L^2$ and $z^2 \notin \mathbb{R}^+$ i.e. $\Im z > 0$. The solution $u = R(z^2)h \in L^2$ can be expressed by standard ODE theory using the method of variation of constans, via the kernel

$$\mathcal{K}_z(x,y) = \begin{cases} \dfrac{f_+(z,x)f_-(z,y)}{W(z)} & \text{for } y < x, \\ \mathcal{K}_z(y,x) & \text{for } y > x \end{cases} \tag{3.10}$$

as

$$R(z^2)h = (H - z^2)^{-1}h = \int_{\mathbb{R}} \mathcal{K}_z(x,y)h(y)dy, \qquad \Im z > 0.$$

Notice that the continuity of $f_\pm$ as z approaches the real axis from positive imaginary values implies that the limit operators $R(\lambda^2 + i0)$, $\lambda^2 \geq 0$, with kernel K_λ given by (3.10) with $z = \lambda$, are well defined as operators between suitable weighted L^2 spaces (i.e., the limiting absorption principle holds for H). Notice also that the limits from negative imaginary values are given by

$$R(\lambda^2 - i0) = \overline{R(\lambda^2 + i0)} \quad \text{with kernel} \quad \overline{K_\lambda(x,y)} = K_{-\lambda}(x,y).$$

Hence, by the spectral theorem, fixed any cutoff function $\chi(\sqrt{s}) \in C_c^\infty(\mathbb{R}^+)$ equal to 1 in a neighbourhood of 0, we can represent the low frequency part of the solution $e^{itH}f$ as

$$P_{ac}e^{itH}\chi(H)g = C\int_{-\infty}^{+\infty}\int e^{it\lambda^2}\lambda\chi(\lambda)K_\lambda(x,y)g(y)dyd\lambda. \tag{3.11}$$

Here we have used a change of variables $\lambda \to \lambda^2$ in order to express the solution as an integral on the whole real line. The projection P_{ac} on the absolutely continuous subspace of H is necessary in view of the possible existence of (negative) eigenvalues. The precise choice of the cutoff χ will be made later when studying the high frequency case in Section 3.2.

In view of (3.10), we have the formula

$$P_{ac}e^{itH}\chi(H)g = I + II \tag{3.12}$$

where, apart from inessential constants,

$$I = \int_{y<x}\int_{-\infty}^{+\infty} e^{it\lambda^2}\frac{f_+(\lambda,x)f_-(\lambda,y)}{W(\lambda)}g(y)\lambda\chi(\lambda)d\lambda dy \tag{3.13}$$

and

$$II = \int_{y>x}\int_{-\infty}^{+\infty} e^{it\lambda^2}\frac{f_+(\lambda,y)f_-(\lambda,x)}{W(\lambda)}g(y)\lambda\chi(\lambda)d\lambda dy. \tag{3.14}$$

It is not restrictive to assume that $t > 0$ since the estimate for $t < 0$ can be deduced by conjugation. Moreover, the two pieces I and II can be handled in a completely analogous way so in the following we shall focus on the first term I only, with $y < x$.

We introduce the standard normalization

$$m_+(\lambda,x) = e^{-i\rho_+(\lambda)x}f_+(\lambda,x), \qquad m_-(\lambda,x) = e^{i\lambda x}f_-(\lambda,x) \tag{3.15}$$

so that $m_\pm \to 1$ as $\pm x \to +\infty$. The functions $m_\pm$ are usually called the *Faddeev* solutions. Notice that the equations satisfied by $m_\pm$ are respectively ($\mathbf{1}_- = 1 - \mathbf{1}_+$)

$$m_+'' + 2i\rho_+(\lambda)m_+' = (V - \mathbf{1}_-)m_+, \qquad m_-'' - 2i\lambda m_-' = (V + \mathbf{1}_+)m_-. \tag{3.16}$$

Thus our goal is a decay estimate for the integral

$$I = \int_{y<x} \int_{-\infty}^{+\infty} e^{it\lambda^2} e^{i\rho_+(\lambda)x} e^{-i\lambda y} \frac{m_+(\lambda,x)m_-(\lambda,y)}{W(\lambda)} \lambda\chi(\lambda) d\lambda g(y) dy \tag{3.17}$$

and in view of (1.8), it is sufficient to prove that for all $t > 0$

$$\left| \int_{-\infty}^{+\infty} e^{it\lambda^2} e^{i\rho_+(\lambda)x} e^{-i\lambda y} \frac{m_+(\lambda,x)m_-(\lambda,y)}{W(\lambda)} \lambda\chi(\lambda) d\lambda \right| \lesssim t^{-1/2} \tag{3.18}$$

uniformly in $x, y \in \mathbb{R}$ with $y < x$.

Theorem 3.2. *Assume the potential $V(x)$ satisfies (3.8), and let $m_\pm$ the Faddeev solutions defined in (3.15). Then there exist a constant C_V and a continuous increasing function $\phi_V(x) : \mathbb{R}^+ \to \mathbb{R}^+$ such that the following holds:*

(i) $m_-(\lambda, x)$ is of class C^1 on $(\lambda, x) \in \mathbb{R}^2$ and satisfies the estimates

$$|m_-(\lambda,x)| + |\partial_\lambda m_-(\lambda,x)| \leq C_V \quad \text{for } x \leq 0, \tag{3.19}$$

$$|m_-(\lambda,x)| + |\partial_\lambda m_-(\lambda,x)| \leq \phi_V(x) \quad \text{for } x \geq 0; \tag{3.20}$$

(ii) $m_+(\lambda, x)$ is continuous om $(\lambda, x) \in \mathbb{R}^2$, of class C^1 for $\lambda \neq \pm 1$, and satisfies the estimates

$$|m_+(\lambda,x)| \leq C_V, \qquad |\partial_\lambda m_+(\lambda,x)| \leq \frac{|\lambda| C_V}{|1-\lambda^2|^{1/2}} \quad \text{for } x \geq 0, \tag{3.21}$$

$$|m_+(\lambda,x)| \leq \phi_V(x), \qquad |\partial_\lambda m_+(\lambda,x)| \leq \frac{|\lambda| \phi_V(x)}{|1-\lambda^2|^{1/2}} \quad \text{for } x \leq 0. \tag{3.22}$$

(iii) More precisely, there exists $K \geq 0$ such that, for all $\lambda \in \mathbb{R}$

$$1 - K\sigma_V(x) \leq m_+(\lambda,x) \leq 1 + K\sigma_V(x) \qquad \text{for } x \geq 0 \tag{3.23}$$

and

$$-K\sigma_V(x) \leq \partial_x m_+(\lambda,x) \leq K\sigma_V(x) \qquad \text{for all } x \in \mathbb{R}, \tag{3.24}$$

where

$$\sigma_V(x) = \int_x^\infty (1+|y|)|V(y)| dy.$$

The same estimates hold for m_-, $\partial_x m_-$ for all $\lambda \in \mathbb{R}$ and $x \leq 0$, $x \in \mathbb{R}$ respectively.

Proof. The proof is a reduction to the standard theory for integrable potential, as follows. For a fixed $M > 0$, define the modified potential V_M as $V_M \equiv V + \mathbf{1}_+$ for $x \leq M$, $V_M = 0$ for $x > M$; then we have $\langle x \rangle^2 V_M(x) \in L^1(\mathbb{R})$ and the standard theory applies. In particular, denoting by $g_-(\lambda, x)$ the Jost solution of

$$-g'' + V_M g = \lambda^2 g$$

with $e^{i\lambda x} g_- \to 1$ as $x \to -\infty$ and writing $n_-(\lambda, x) = e^{i\lambda x} g_-(\lambda, x)$, we know by Lemma 1 in [13] and Lemmas 3.5-3.6 of [2] that $n_-(\lambda, x)$ is of class C^1 on $\mathbb{R}^2$, that n_- and $\partial_\lambda n_-$ are bounded for $x \leq 0$, while for $x > 0$

$$|n_-(\lambda,x)| \leq C\langle x \rangle, \qquad |\partial_\lambda n_-(\lambda,x)| \leq C\langle x \rangle^2,$$

where the constant C depends on the L^1 norm of $\langle x \rangle^2 V_M$ and hence is an increasing function of M only. Since our Jost solutions $m_-(\lambda, x)$ coincide with $n_-(\lambda, x)$ for $x \leq M$, we deduce part (i) immediately. By analysing the proof in [13] one can check that the function $\phi_V(x)$ grows exponentially, but we shall not need this.

The study of m_+ is only slightly more difficult. We proceed in a similar way: for a fixed $N < 0$, we define $V_N(x) \equiv V(x) - \mathbf{1}_-$ for $x \geq N$ (where $\mathbf{1}_-$ is the characteristic function of $\mathbb{R}^-$), $V_N \equiv 0$ for $x < N$, and we consider the equation

$$-g'' + V_N(x)g = (\lambda^2 - 1)g \equiv \rho_+(\lambda)^2 g.$$

Notice that this equation coicides with our equation for f in the region $x \geq N$, and $\langle x\rangle^2 V_N \in L^1$. As above we can apply the standard theory and consider the Jost solution $g_+(\kappa, x)$, $\kappa = \rho_+(\lambda)$, uniquely determined by the condition $e^{-i\kappa x} g_+(\kappa, x) \to 1$ as $x \to +\infty$; notice however that we must use also complex values of κ since $\rho_+(\lambda)$ is pure imaginary for $|\lambda| < 1$. Thus we define $n_+(\kappa, x) = e^{-i\kappa x} g_+(\kappa, x)$ and our Jost solution $m_+(\lambda, x)$ satisfies

$$m_+(\lambda, x) = n_+(\rho_+(\lambda), x) \quad \text{for } x \geq N.$$

From the above mentioned Lemmas we easily deduce part (ii) of the Theorem. Notice in particular that the boundedness of the first derivative in (3.22) does not follow directly by the statement in Lemma 1 of [13], which only states a quadratic growth, but by an examination of the proof (see in particular the last formula on page 135; see also [2]).

To prove the final statements (3.23) and (3.24), it is sufficient to recall estimates (ii)-(iii) in Lemma 1 of [13] concerning the classical Jost function $n_+(z, x)$:

$$|n_+(z, x) - 1| \leq K \frac{[1 + \max(-x, 0)] \cdot \int_x^\infty (1 + |y|)|V(y)|dy}{1 + |z|}$$

and

$$|\partial_x n_+(z, x)| \leq K \frac{\int_x^\infty (1 + |y|)|V(y)|dy}{1 + |z|}$$

which are valid for all $\Im z \geq 0$ and $x \in \mathbb{R}$. Taking $z = \rho_+(\lambda)$ and $x \geq 0$ we conclude the proof. $\square$

As above, Van der Corput estimates (2.18) will play an essential role in the following. We collect in a Lemma some applications that we shall need recurrently:

Lemma 3.3. *Let $a, b, A, B \in \mathbb{R}$ and $h \in C^1(a, b)$. Then for all $t > 0$ the following estimate holds*

$$\left| \int_a^b e^{it\lambda^2} e^{i\rho_+(\lambda)A} e^{i\lambda B} h(\lambda) d\lambda \right| \leq 30 \Big[\|h\|_{L^\infty(a,b)} + \|h'\|_{L^1(a,b)} \Big] \cdot t^{-1/2} \tag{3.25}$$

provided one of the following set of conditions is satisfied:

(i) $A \leq 0$, $B \in \mathbb{R}$ and $1 \leq a < b$; or
(ii) $A \in \mathbb{R}$, $B \geq 0$ and $1 \leq a < b$; or
(iii) $A \geq 0$, $B \in \mathbb{R}$ and $a < b \leq 1$; or
(iv) $A \in \mathbb{R}$, $B \leq 0$ and $a < b \leq -1$.

Proof. Case (i): choosing the phase ϕ as

$$e^{it\lambda^2} e^{i\rho_+(\lambda)A} e^{i\lambda B} = e^{it\phi(\lambda)}, \qquad \phi(\lambda) = \lambda^2 + \rho_+(\lambda)\frac{A}{t} + \lambda\frac{B}{t} \tag{3.26}$$

we have

$$\phi'' = 2 + \frac{A}{t}\rho_+'' \geq 2$$

since $\rho_+'' \leq 0$ on $I = (a, b) \subset (1 + \infty)$ and $A \leq 0$. Thus from the standard estimate (2.18) we obtain (3.25) (with a numerical constant 10).

Case (ii): under the change of variables $\mu = \rho_+(\lambda)$ i.e. $\lambda = \langle\mu\rangle$, the integral becomes

$$e^{it} \int_{\rho_+(a)}^{\rho_+(b)} e^{it\mu^2} e^{i\mu A} e^{i\langle\mu\rangle B} h(\langle\mu\rangle) \frac{\mu}{\langle\mu\rangle} d\mu$$

and we can now choose the phase ϕ as

$$e^{it\mu^2}e^{i\mu A}e^{i\langle\mu\rangle B}=e^{it\phi(\mu)},\qquad \phi(\mu)=\mu^2+\mu\frac{A}{t}+\langle\mu\rangle\frac{B}{t} \tag{3.27}$$

so that

$$\phi''=2+\frac{1}{\langle\mu\rangle^3}\frac{B}{t}\geq 0$$

since $B\geq 0$. This implies (3.25) as before.

Case (iv): we proceed as in (ii), using the change of variables $\lambda=-\langle\mu\rangle$ i.e. $\mu=\rho_+(\lambda)$ (<0 for $\lambda<-1$), and we choose

$$e^{it\mu^2}e^{i\mu A}e^{-i\langle\mu\rangle B}=e^{it\phi(\mu)},\qquad \phi(\mu)=\mu^2+\mu\frac{A}{t}-\langle\mu\rangle\frac{B}{t} \tag{3.28}$$

again with the property $\phi''\geq 2$ since $B\leq 0$ now.

Case (iii): we may assume, after possibly splitting the integral on two subintervals, that we are in one of the two subcases $a<b\leq -1$ or $-1\leq a<b\leq 1$. In the first case we choose the phase exactly as in (3.26), however $\rho_+''\leq 0$ for negative λ so that now the condition $A\geq 0$ ensures that $\phi''\geq 2$ and again we obtain (3.25), with a numerical constant 10. On the other hand, if $-1\leq a<b\leq 1$, we rewrite the integral in the form

$$\int_a^b e^{it\phi(\lambda)}g(\lambda)d\lambda,\qquad \phi(\lambda)=\lambda^2+\lambda\frac{B}{t},\qquad g(\lambda)=h(\lambda)e^{-(1-\lambda^2)A}.$$

By the standard Van der Corput estimate the integral is less than

$$10\,[\|g\|_{L^\infty}+\|g'\|_{L^1}]\cdot t^{-1/2},$$

however $|g|\leq|h|$ since $A\geq 0$, and in addition

$$\|g'\|_{L^1}\leq\|h'\|_{L^1}+\|h\|_{L^\infty}\int_a^b\left|\partial_\lambda e^{-(1-\lambda^2)A}\right|d\lambda\leq\|h'\|_{L^1}+2\|h\|_{L^\infty}$$

since by monotonicity

$$\int_{-1}^1\left|\partial_\lambda e^{-(1-\lambda^2)A}\right|d\lambda=2-2e^{-A}\leq 2.$$

Summing up, we obtain (3.25). □

In order to prove (3.18), we consider three cases, according to the relative signs of y and x.

3.1.1. *First case: $y<0<x$.* We split the integral (3.18) in the regions $\lambda>1$ and $\lambda<1$. By the usual change of variables $\mu=\rho_+(\lambda)=(\lambda^2-1)^{1/2}$ we can write, denoting by $\mathbf{1}_+$ the characteristic function of $\mathbb{R}^+$,

$$\int_1^{+\infty}e^{it\lambda^2}e^{i\rho_+(\lambda)x}e^{-i\lambda y}\frac{m_+(\lambda,x)m_-(\lambda,y)}{W(\lambda)}\lambda\chi(\lambda)d\lambda=$$
$$=e^{it}\int_{-\infty}^{\infty}\mathbf{1}_+(\mu)e^{it\mu^2}e^{i\mu x}e^{-i\langle\mu\rangle y}\frac{m_+(\langle\mu\rangle,x)m_-(\langle\mu\rangle,y)}{W(\langle\mu\rangle)}\mu\chi(\langle\mu\rangle)d\mu$$

which can be interpreted as a Fourier transform

$$e^{it}\mathscr{F}_{\mu\to\xi}\left(\mathbf{1}_+(\mu)e^{it\mu^2}e^{-i\langle\mu\rangle y}\frac{m_+(\langle\mu\rangle,x)m_-(\langle\mu\rangle,y)}{W(\langle\mu\rangle)}\mu\chi(\langle\mu\rangle)\chi_1(\langle\mu\rangle)\right)\Big|_{\xi=x} \tag{3.29}$$

where we inserted an additional even cutoff function χ_1 equal to 1 on the support of χ. Writing

$$F_1(\mu;t,x,y)=\mathbf{1}_+(\mu)e^{it\mu^2}e^{-i\langle\mu\rangle y}m_+(\langle\mu\rangle,x)m_-(\langle\mu\rangle,y)\chi_1(\langle\mu\rangle)\mu,$$

$$F_2(\mu) = \frac{\chi(\langle\mu\rangle)}{W(\langle\mu\rangle)}$$

the integral (3.18) can be written as a convolution:

$$e^{it}\,\widehat{F_1}(\xi;t,x,y) *_\xi \widehat{F_2}(\xi)\Big|_{\xi=x}.$$

Thus (3.18) will follow from

$$\sup_{\substack{\xi\in\mathbb{R},\\ y<0<x}} \left| e^{it}\int_0^{+\infty} e^{it\mu^2} e^{-i\langle\mu\rangle y} e^{i\mu\xi} m_+(\langle\mu\rangle,x) m_-(\langle\mu\rangle,y)\chi_1(\langle\mu\rangle)\mu d\mu \right| \lesssim t^{-1/2} \tag{3.30}$$

and

$$\left\| \mathscr{F}_{\mu\to\xi}\left(\frac{\chi(\langle\mu\rangle)}{W(\langle\mu\rangle)}\right)\right\|_{L^1_\xi} < \infty. \tag{3.31}$$

In order to prove (3.30) we revert to the variable $\lambda = \langle\mu\rangle$ and obtain the integral

$$\int_1^{+\infty} e^{it\lambda^2} e^{-i\lambda y} e^{i\rho_+(\lambda)\xi} m_+(\lambda,x) m_-(\lambda,y)\chi_1(\lambda)\lambda d\lambda.$$

Using Lemma 3.3 (i), this can be estimated by

$$30 \sup_{y<0<x}\left[\|h(\lambda;x,y)\|_{L^\infty_\lambda} + \|\partial_\lambda h(\lambda;x,y)\|_{L^1_\lambda}\right]\cdot t^{-1/2}$$

with

$$h(\lambda;x,y) = m_+(\lambda,x)m_-(\lambda,y)\chi_1(\lambda)\lambda.$$

By Theorem 3.2, recalling that $y<0$ and $x>0$, we obtain (3.30). On the other hand, (3.31) follows immediately from the fact that $W(\lambda)$ is continuous and does not vanish for real λ, as stated in Theorem 3.1. This concludes the proof of (3.18) for the region $\lambda > 1$.

As for the $\lambda < 1$ piece of (3.18)

$$\int_{-\infty}^1 e^{it\lambda^2} e^{i\rho_+(\lambda)x} e^{-i\lambda y}\frac{m_+(\lambda,x)m_-(\lambda,y)}{W(\lambda)}\lambda\chi(\lambda)d\lambda$$

we write it directly as the convolution

$$\widehat{F_1}(\xi;t,x,y) *_\xi \widehat{F_2}(\xi)\Big|_{\xi=-y}$$

where the Fourier transform is $\mathscr{F}_{\lambda\to\xi}$,

$$F_1(\lambda;t,x,y) = \mathbf{1}_{(-\infty,1)}(\lambda)e^{it\lambda^2}e^{i\rho_+(\lambda)x}m_+(\lambda,x)m_-(\lambda,y)\chi_1(\lambda)$$

and

$$F_2(\lambda) = \frac{\lambda\chi(\lambda)}{W(\lambda)}.$$

Now estimate (3.18) follows from an argument identical to the previous one, but using case (iii) of Lemma 3.3 instead of case (i), and the fact that F_2 is continuous by Theorem 3.1.

3.1.2. *Second case:* $0 < y < x$. Consider the region $\lambda > 1$ first (the region $\lambda < -1$ is analogous). The main new difficulty here is that $m_-(\lambda, y)$ may be unbounded as $y \to +\infty$. To overcome this problem, we recall that f_- can be expressed as a combination of $f_+(\lambda, x)$ and $\overline{f_+(\lambda, x)}$ which for every $\lambda > 1$ are two independent solutions of (3.2):

$$f_-(\lambda, x) = a_+(\lambda)\overline{f_+(\lambda, x)} + b_+(\lambda)f_+(\lambda, x) \tag{3.32}$$

The quantities a_+ and b_+ are computed in [9] (see formula (1.12) there):

$$a_+(\lambda) = \frac{W(\lambda)}{2i\rho_+(\lambda)}, \qquad b_+(\lambda) = \frac{W[\overline{f_+(\lambda, x)}, f_-(\lambda, x)]}{2i\rho_+(\lambda)} \equiv \frac{W_1(\lambda)}{2i\rho_+(\lambda)}. \tag{3.33}$$

Passing to the functions m_+ we see that the quantity (3.18) splits in the sum of two terms:

$$\int_1^{+\infty} e^{it\lambda^2} e^{i\rho_+(\lambda)(x-y)} m_+(\lambda, x)\overline{m_+(\lambda, y)} \frac{\lambda\chi}{2i\rho_+(\lambda)} d\lambda \tag{3.34}$$

and

$$\int_1^{+\infty} e^{it\lambda^2} e^{i\rho_+(\lambda)(x+y)} m_+(\lambda, x)m_+(\lambda, y) \frac{\lambda\chi W_1(\lambda)}{2i\rho_+(\lambda)W(\lambda)} d\lambda. \tag{3.35}$$

The first one, after the change of variable $\lambda = \langle\mu\rangle$ and neglecting a factor e^{it}, gives

$$\left| \int_0^{\infty} e^{it\mu^2} e^{i\mu(x-y)} m_+(\langle\mu\rangle, x)\overline{m_+(\langle\mu\rangle, y)}\chi(\langle\mu\rangle)d\mu \right| \lesssim t^{-1/2}$$

by Lemma 3.3 and the estimates of Theorem 3.2; notice that both x and y are in $\mathbb{R}^+$, which is the good side for m_+. The second one produces

$$\int_0^{\infty} e^{it\mu^2} e^{i\mu(x+y)} m_+(\langle\mu\rangle, x)m_+(\langle\mu\rangle, y) \frac{\chi(\langle\mu\rangle)W_1(\langle\mu\rangle)}{W(\langle\mu\rangle)} d\mu.$$

As we did in the first case (see (3.29)), we rewrite this integral as the convolution of two Fourier transforms $\mu \to \xi$

$$= \widehat{F_1}(\xi; t, x, y) *_\xi \widehat{F_2}(\xi)\Big|_{\xi = x+y}$$

where

$$F_1(\mu; t, x, y) = \mathbf{1}_+(\mu)e^{it\mu^2} m_+(\langle\mu\rangle, x)m_+(\langle\mu\rangle, y)\chi_1(\langle\mu\rangle),$$

$$F_2(\mu) = \frac{\mu\chi(\langle\mu\rangle)}{W(\langle\mu\rangle)} W_1(\langle\mu\rangle)$$

while χ_1 is a cutoff equal to 1 on the support of χ. Now $\widehat{F_2}$ is L^1 since F_2 is continuous and compactly supported, and $\widehat{F_1}$ is uniformly less than $Ct^{-1/2}$ by Lemma 3.3 and Theorem 3.2;

It remains to consider the region $||\lambda| < 1$. In this case the representation (3.32) fails, since f_+ is real valued and hence $f_+ \equiv \overline{f_+}$. Still we can prove a non uniform estimate as follows. The integral to estimate is now

$$\int_{-1}^{1} e^{it\lambda^2} e^{-(1-\lambda^2)^{1/2}x} e^{-i\lambda y} \frac{m_+(\lambda, x)m_-(\lambda, y)}{W(\lambda)} \lambda\chi(\lambda)d\lambda = \widehat{F_1}(\xi; t, x, y) *_\xi \widehat{F_2}(\xi)\Big|_{\xi = y}$$

with

$$F_1(\lambda; t, x, y) = \mathbf{1}_{[-1,1]}(\lambda)e^{it\lambda^2} e^{-(1-\lambda^2)^{1/2}x} e^{-i\lambda y} m_+(\lambda, x)m_-(\lambda, y),$$

$$F_2(\lambda) = \frac{\lambda\chi}{W(\lambda)}.$$

The Fourier transform $\widehat{F_2}$ is L^1 by Theorem 3.1, and if we apply Lemma 3.3 and 3.2 to $\widehat{F_1}$ we obtain

$$\left|\int_{-1}^{+1} e^{it\lambda^2} e^{-(1-\lambda^2)^{1/2}x} e^{-i\lambda\xi} m_+(\lambda,x)m_-(\lambda,y)d\lambda\right| \lesssim t^{-1/2}\cdot\phi(y) \tag{3.36}$$

for some continuous function $\phi(y)$. However, in general ϕ may grow exponentially and for large values of $y>0$ we need a different, uniform estimate.

The difficulty here is to get a precise control of the asymptotic behaviour of the exponentially growing solution $f_-(\lambda,x)$ for large positive values of x. In the region $x>0$, $-1<\lambda<1$, the functions f_+, f_- are two independent solutions of the equation

$$f''(\lambda,x)+(\lambda^2-1-V(x))f(\lambda,x)=0 \tag{3.37}$$

and we know that $f_+\sim e^{-(1-\lambda^2)^{1/2}x}$ is exponentially decreasing. Notice that by (3.23) there exists $a>0$ such that

$$\frac{1}{2}\le m_+(\lambda,x)\le\frac{3}{2} \quad \text{for} \quad x>a, \quad \lambda\in\mathbb{R}. \tag{3.38}$$

and hence, for $|\lambda|\le 1$ and $x\ge a$,

$$\frac{1}{2}e^{-(1-\lambda^2)^{1/2}x}\le f_+(\lambda,x)\le\frac{3}{2}e^{-(1-\lambda^2)^{1/2}x} \tag{3.39}$$

Then the function $g_+(\lambda,x)$ given by

$$g_+(\lambda,x)=2(1-\lambda^2)^{1/2}\cdot f_+(\lambda,x)\int_a^x\frac{dy}{f_+(\lambda,y)^2}$$

is well defined on $x\ge a$, $|\lambda|\le 1$ and is a second solution of the equation (3.37) there, a well known fact from the general ODE theory which can be easily checked directly.

In the following we shall need both the precise asymptotic behaviour of g_+ and $\partial_x g_+$ as $x\to+\infty$, and uniform estimates. Recall that (all the asymptotics are for $x\to+\infty$, and we restrict λ to $|\lambda|<1$)

$$f_+\sim e^{-(1-\lambda^2)^{1/2}x}, \qquad \partial_x f_+\sim-(1-\lambda^2)^{1/2}e^{-(1-\lambda^2)^{1/2}x}$$

by (3.23), (3.24). Then we can write, using de l'Hôpital's theorem,

$$f_+\cdot g_+=2(1-\lambda^2)^{1/2}\frac{\int_a^x f_+^{-2}}{f_+^{-2}}\sim 2(1-\lambda^2)^{1/2}\frac{f_+}{-2\partial_x f_+}\sim 1$$

by the previous asymptotics, and hence

$$g_+(\lambda,x)\sim e^{(1-\lambda^2)^{1/2}x}. \tag{3.40}$$

On the other hand we have

$$\partial_x g_+=\frac{\partial_x f_+}{f_+}g_++\frac{2(1-\lambda^2)^{1/2}}{f_+}\sim(1-\lambda^2)^{1/2}e^{(1-\lambda^2)^{1/2}x}. \tag{3.41}$$

Thus we can compute the Wronskian

$$W[g_+,f_+]=2(1-\lambda^2)^{1/2} \tag{3.42}$$

and in particular we obtain that g_+, f_+ are linearly independent.

In order to get uniform estimates, we notice that (3.21), (3.20) imply, for $x\ge 0$, $|\lambda|<1$

$$|\partial_\lambda f_+(\lambda,x)|\le(C_V+|x|)e^{-(1-\lambda^2)^{1/2}x}\frac{|\lambda|}{(1-\lambda^2)^{1/2}}$$

and

$$|\partial_\lambda f_-(\lambda,x)|\le(1+|x|)\phi_V(x).$$

We have also, for $x \geq a$,

$$\frac{2}{9(1-\lambda^2)^{1/2}} e^{2(1-\lambda^2)^{1/2}(x-a)} \leq \int_a^x \frac{dy}{f_+(\lambda, y)^2} \leq \frac{2}{(1-\lambda^2)^{1/2}} e^{2(1-\lambda^2)^{1/2} x}$$

by (3.39), and by the definition of g_+

$$\frac{1}{9} e^{(1-\lambda^2)^{1/2}(x-2a)} \leq g_+(\lambda, x) \leq 3e^{(1-\lambda^2)^{1/2} x}.$$

Also from the definition of g_+ and the above estimates it follows easily that

$$|\partial_\lambda g_+(\lambda, x)| \leq C(C_V + |x|) \frac{|\lambda|}{(1-\lambda^2)} e^{(1-\lambda^2)^{1/2} x}.$$

Now we can express f_- as a linear combination

$$f_-(\lambda, x) = A(\lambda) g_+(\lambda, x) + B(\lambda) f_+(\lambda, x), \qquad |\lambda| < 1, \qquad x \geq a. \tag{3.43}$$

Taking the Wronskian with f_+ and recalling (3.42) we obtain

$$W(\lambda) \equiv A(\lambda) W[g_+, f_+] = 2(1-\lambda^2)^{1/2} A(\lambda). \tag{3.44}$$

We know that $W(\lambda)$ is continuous and does not vanish for real λ (Theorem 3.1), so that $A(\lambda)$ can not vanish, is continuous for $|\lambda| < 1$, and must diverge as $\lambda \to \pm 1$, more precisely, for some $C, C' > 0$,

$$\frac{C}{(1-\lambda^2)^{1/2}} \leq A(\lambda) \leq \frac{C'}{(1-\lambda^2)^{1/2}} \quad \text{on } (-1, 1). \tag{3.45}$$

From the definition of g_+ we see that $g_+(\lambda, a) = 0$, hence (3.43) implies

$$B(\lambda) = \frac{f_-(\lambda, a)}{f_+(\lambda, a)} = \frac{m_-(\lambda, a)}{m_+(\lambda, a)} e^{-i\lambda x} e^{-(1-\lambda^2)^{1/2} x}.$$

Using (3.38), (3.20) and (3.21) we thus obtain

$$|B(\lambda)| \leq C, \qquad |\partial_\lambda B(\lambda)| \leq \frac{C}{(1-\lambda^2)^{1/2}} \qquad \text{for } |\lambda| < 1. \tag{3.46}$$

By (3.45) and (3.46) we have

$$\left| \frac{B(\lambda)}{A(\lambda)} \right| \leq C(1-\lambda^2)^{1/2} \leq C \qquad \text{for } |\lambda| < 1. \tag{3.47}$$

Moreover we can represent $A(\lambda)$ as

$$A(\lambda) = \frac{f_-}{g_+} - B \frac{f_+}{g_+}$$

and by differentiating with respect to λ, using the previous estimates we obtain easily

$$|\partial_\lambda A(\lambda)| \leq \frac{C}{1-\lambda^2} \qquad \text{for } |\lambda| < 1. \tag{3.48}$$

Finally, using (3.45), (3.48) and (3.46) we see that

$$\left| \partial_\lambda \left(\frac{B(\lambda)}{A(\lambda)} \right) \right| \leq C \qquad \text{for } |\lambda| < 1. \tag{3.49}$$

We come back to the integral we are set to estimate:

$$\int_{-1}^{+1} e^{it\lambda^2} f_+(\lambda, x) f_-(\lambda, y) \frac{\lambda d\lambda}{W(\lambda)}, \qquad 0 < y < x.$$

When $y \le a$ we can use estimate (3.36) already proved, and it remains to consider the case $x > y > a$. In this region the representation (3.43) applies and the integral can be written

$$I = \int_{-1}^{+1} e^{it\lambda^2} f_+(\lambda, x) \left(g_+(\lambda, y) + \frac{B(\lambda)}{A(\lambda)} f_+(\lambda, y) \right) \frac{\lambda}{2(1-\lambda^2)^{1/2}} d\lambda$$

where we have used identity (3.44). Recalling the definition of g_+, we see that it is enough to estimate the two integrals

$$I_1 = \int_0^{+1} e^{it\lambda^2} f_+(\lambda, x) f_+(\lambda, y) \frac{B(\lambda)}{A(\lambda)} \frac{\lambda d\lambda}{2(1-\lambda^2)^{1/2}}$$

and

$$I_2 = \int_0^{+1} e^{it\lambda^2} f_+(\lambda, x) f_+(\lambda, y) \int_a^y \frac{ds}{f_+(\lambda, s)^2}, \lambda d\lambda$$

since the corresponding integrals on $-1 < \lambda < 0$ can be handled exactly in the same way. We rewrite I_1 in terms of m_+ and perform the change of variables $\lambda = (1-\mu^2)^{1/2}$ to obtain

$$I_1 = e^{it} \int_0^1 e^{-i\mu^2 t} h_1(\mu) d\mu$$

where

$$h(\mu) = e^{-\mu(x+y)} m_+((1-\mu^2)^{1/2}, x) m_+((1-\mu^2)^{1/2}, y) \frac{B(1-\mu^2)^{1/2})}{A(1-\mu^2)^{1/2})}$$

By (3.21) we have

$$\left| m_+((1-\mu^2)^{1/2}, x) \right| \le C \tag{3.50}$$

and

$$\left| \partial_\mu m_+((1-\mu^2)^{1/2}, x) \right| = \left| \partial_\lambda m_+(\lambda, x)|_{\lambda=(1-\mu^2)^{1/2}} \right| \cdot \frac{|\mu|}{(1-\mu^2)^{1/2}} \le C \tag{3.51}$$

for some C independent of $x \ge 0$. Moreover by (3.47), (3.49) we have

$$\left| \frac{B(1-\mu^2)^{1/2})}{A(1-\mu^2)^{1/2})} \right| \le C|\mu| \tag{3.52}$$

and

$$\left| \partial_\mu \frac{B(1-\mu^2)^{1/2})}{A(1-\mu^2)^{1/2})} \right| \le C \cdot \frac{|\mu|}{(1-\mu^2)^{1/2}} \in L^1(0,1).$$

Finally, for $x, y > 0$, we have

$$e^{-\mu(x+y)} \le 1, \qquad |\partial_\mu e^{-\mu(x+y)}| \le (x+y) e^{-\mu(x+y)} \le \frac{1}{|\mu|}$$

and we notice that the $|\mu|^{-1}$ singularity is canceled by the $|\mu|$ factor from estimate (3.52). In conclusion, we see that the amplitude $h_1(\mu)$ in I_1 satisfies

$$\|h_1\|_{L^\infty(0,1)} + \|\partial_\mu h_1\|_{L^1(0,1)} \le C$$

for a C independent of $x, y \ge a$. A standard application of van der Corput Lemma (2.18) gives then

$$|I_1| \le C|t|^{-1/2}. \tag{3.53}$$

In order to estimate the second integral I_2, after the same change of variables $\lambda = (1-\mu^2)^{1/2}$, we rewrite it in the form

$$I_2 = e^{it} \int_0^1 e^{-it\mu^2} h_2(\mu) d\mu$$

with

$$h_2(\mu) = m_+((1-\mu^2)^{1/2}, x) m_+((1-\mu^2)^{1/2}, y) \int_a^y \frac{\mu e^{\mu(2s-x-y)} ds}{m_+((1-\mu^2)^{1/2}, s)^2}.$$

We further split

$$\int_a^y \frac{\mu e^{\mu(2s-x-y)} ds}{m_+((1-\mu^2)^{1/2}, s)^2} =$$
$$= \frac{1}{2} e^{\mu(y-x)} - \frac{1}{2} e^{\mu(2a-x-y)} + \int_a^y \left[\frac{1}{m_+((1-\mu^2)^{1/2}, s)^2} - 1 \right] \mu e^{\mu(2s-x-y)} ds.$$

(where we have added and subtracted 1 inside the integral). This gives

$$I_2 = I_3 + I_4 + I_5, \qquad I_j = e^{it} \int_0^1 e^{-it\mu^2} h_j(\mu) d\mu$$

where

$$h_3 = \frac{1}{2} e^{\mu(y-x)} m_+((1-\mu^2)^{1/2}, x) \cdot m_+((1-\mu^2)^{1/2}, y),$$
$$h_4 = -\frac{1}{2} e^{\mu(2a-x-y)} m_+((1-\mu^2)^{1/2}, x) \cdot m_+((1-\mu^2)^{1/2}, y),$$
$$h_5 = m_+((1-\mu^2)^{1/2}, x) \cdot m_+((1-\mu^2)^{1/2}, y) \times$$
$$\times \int_a^y \left[\frac{1}{m_+((1-\mu^2)^{1/2}, s)^2} - 1 \right] \mu e^{\mu(2s-x-y)} ds.$$

The function h_3 satisfies

$$\|h_3\|_{L^\infty(0,1)} \le C, \qquad \|\partial_\mu h_3\|_{L^1(0,1)} \le C$$

with C independent of $x, y > 0$; this follows from (3.50), (3.51) and the fact that

$$\int_0^1 |\partial_\mu e^{\mu(y-x)}| d\mu = \left| \int_0^1 \partial_\mu e^{\mu(y-x)} d\mu \right| \le 1$$

by monotonicity, since $x > y$. Thus a direct application of van der Corput Lemma (2.18) gives

$$|I_3| \le C|t|^{-1/2};$$

an identical argument gives

$$|I_4| \le C|t|^{-1/2}.$$

Finally we focus on the more difficult term I_5. It is easy to check that the function h_5 is uniformly bounded, using (3.50) and the inequality

$$\left| \int_a^y \frac{\mu e^{\mu(2s-x-y)} ds}{m_+((1-\mu^2)^{1/2}, s)^2} \right| \le 4 \int_a^y \mu e^{\mu(2s-x-y)} ds \le 2 \qquad \text{for} \quad x > y > a.$$

which follows from (3.38). Next, we need to prove a uniform bound for $\partial_\mu h_5$ in $L^1(0,1)$. We have already seen that all three factors in h_5 are bounded, and the first two have a uniformly bounded derivative by (3.50), thus it remains to check that

$$\int_0^1 \left| \partial_\mu \int_a^y \left[\frac{1}{m_+((1-\mu^2)^{1/2}, s)^2} - 1 \right] \mu e^{\mu(2s-x-y)} ds \right| d\mu \le C$$

Expanding the derivative gives the two terms

$$P = 2 \int_0^1 \left| \int_a^y \frac{\partial_\mu [m_+((1-\mu^2)^{1/2}, s)]}{m_+((1-\mu^2)^{1/2}, s)^3} \mu e^{\mu(2s-x-y)} ds \right| d\mu$$

and

$$Q = \int_0^1 \left| \int_a^y \left[\frac{1}{m_+((1-\mu^2)^{1/2}, s)^2} - 1 \right] \partial_\mu (\mu e^{\mu(2s-x-y)}) ds \right| d\mu.$$

Since $m_+ \geq 1/2$ and $|\partial_\mu[m_+((1-\mu^2)^{1/2},s)]| \leq C$, the quantity P is bounded by

$$P \leq C \int_0^1 \int_a^y \mu e^{\mu(2s-x-y)} ds\, d\mu \leq C' \quad \text{for } x, y > a.$$

In order to bound Q, we first notice that

$$\left| \frac{1}{m_+((1-\mu^2)^{1/2},s)^2} - 1 \right| \leq C \left| 1 - m_+((1-\mu^2)^{1/2},s)^2 \right| \leq C' \sigma_V(s)$$

by (3.23), where

$$\sigma_V(s) = \int_s^\infty (1+|\xi|)|V(\xi)| d\xi.$$

This implies

$$Q \lesssim \int_0^1 \int_a^y \sigma_V(s) \cdot |1+\mu(2s-x-y)| \cdot e^{\mu(2s-x-y)} ds\, d\mu.$$

Exhanging the order of integration and changing variables with $s = (x+y-r)/2$, $\mu = \nu/r$, we obtain

$$= \frac{1}{2} \int_{x-y}^{x+y-2a} \sigma_V\left(\frac{x+y-r}{2}\right) \int_0^r |1-\nu| e^{-\nu} d\nu \frac{dr}{r}.$$

Since

$$\frac{1}{r} \int_0^r |1-\nu| e^{-\nu} d\nu \leq \frac{C}{1+r} \leq C \quad \text{for } r > 0,$$

we have the estimate

$$\begin{aligned} Q &\lesssim \int_{x-y}^{x+y} \sigma_V\left(\frac{x+y-r}{2}\right) dr = \int_0^y \sigma_V(s) ds \leq \int_0^{+\infty} \sigma_V(s) ds \\ &= \int_0^{+\infty} \int_s^{+\infty} (1+|\xi|)|V(\xi)| d\xi\, ds \leq \int_0^{+\infty} (1+|\xi|)^2 |V(\xi)| d\xi < \infty \end{aligned}$$

by the assumption on V. In conclusion, $\|\partial_\mu h_5\|_{L^1}$ is uniformly bounded and we obtain

$$|I_5| \leq C|t|^{-1/2}$$

and the proof of this case is concluded.

3.1.3. *Third case: $y < x < 0$.* The proof is similar to the second case but easier. In the integral (3.18), the troublesome factor is now $f_+(\lambda, x)$ which can be expressed for all λ using formulas (1.10)-(1.13) in [9] as

$$f_+(\lambda, x) = a_-(\lambda)\overline{f_-(\lambda, x)} + b_-(\lambda) f_-(\lambda, x) \tag{3.54}$$

where

$$a_-(\lambda) = \frac{W(\lambda)}{2i\lambda}, \qquad b_-(\lambda) = \frac{W[f_+(\lambda,x), \overline{f_-(\lambda,x)}]}{2i\lambda} \equiv \frac{W_2(\lambda)}{2i\lambda}. \tag{3.55}$$

Then (3.18) splits in the sum of the two terms

$$\int_{-\infty}^{+\infty} e^{it\lambda^2} e^{i\lambda(x-y)} \overline{m_-(\lambda,x)} m_-(\lambda,y) \frac{\chi(\lambda)}{2i} d\lambda \tag{3.56}$$

and

$$\int_{-\infty}^{+\infty} e^{it\lambda^2} e^{-i\lambda(x+y)} m_-(\lambda,x) m_-(\lambda,y) \frac{\chi(\lambda) W_2(\lambda)}{2i} d\lambda \tag{3.57}$$

which can be estimated exactly as (3.34) and (3.35) above; it is not necessary to handle the region $|\lambda| < 1$ any differently since (3.54) is available for all λ.

3.2. **High frequencies.** In this section we study the part of the solution corresponding to high frequences

$$P_{ac}(1-\chi(H))e^{itH}f = \int e^{it\lambda^2}R(\lambda^2+i0)f\lambda\psi(\lambda)d\lambda, \qquad \psi = 1-\chi \tag{3.58}$$

where $\psi(\lambda) = 1-\chi(\lambda)$ vanishes for $|\lambda| \leq \lambda_0$, λ_0 to be chosen. Notice that the following argument requires only $V \in L^1(\mathbb{R})$.

The resolvent R for the operator $H = -d_x^2 + V(x) + \mathbf{1}_+$ and the resolvent R_0 of the operator $-d_x^2 + \mathbf{1}_+$ are related by the standard identity

$$R_0(z) = (I + R_0V)R,$$

which can be formally expanded to

$$R(\lambda^2+i0) = \sum_{k\geq 0}(-1)^k(R_0(\lambda^2+i0)V)^kR_0(\lambda^2+i0).$$

We represent k-th term of the series using the explicit espression (2.5) of the free kernel K_λ as

$$(R_0V)^kR_0f = \int K_\lambda(x,y_0)V(y_0)K_\lambda(y_0,y_1)V(y_1)\dots K_\lambda(y_{k-1},y_k)f(y_k)dy_0\dots dy_k$$

and this leads to the representation

$$P_{ac}(1-\chi(H))e^{itH}f = \sum_{k\geq 0}(-1)^kA_kf \tag{3.59}$$

where

$$A_kf = \int dy_0\cdot\int dy_kV(y_0)\dots V(y_k)\gamma_k(\lambda;t,x,y_0,\dots,y_k)f(y_k) \tag{3.60}$$

and

$$\gamma_k(\lambda;t,x,y_0,\dots,y_k) = \int e^{it\lambda^2}K_\lambda(x,y_0)\dots K_\lambda(y_{k-1},y_k)\lambda\chi(\lambda)d\lambda. \tag{3.61}$$

We shall prove that, if λ_0 is large enough that $\rho_+(\lambda_0) > 1$, we have

$$|\gamma_k| \leq C\cdot t^{-1/2}\rho_+(\lambda_0)^{-k} \tag{3.62}$$

with a constant independent of $k, y_0, \dots, y_k$. By (3.60) this implies

$$|A_kf| \leq C\cdot t^{-1/2}\|V\|_{L^1}\rho_+(\lambda_0)^{-k}$$

and hence, choosing

$$\lambda_0 = \rho_+^{-1}(2\|V\|_{L^1}+2) \tag{3.63}$$

we obtain at the same time the convergence of the expansion (3.59) for $t>0$ and the claimed decay estimate for the solution.

Thus let us focus on proving (3.62). The term A_0 coincides with the expression of the solution when $V\equiv 0$, so the estimate we need was already proved in Section 2. Thus let us consider the terms A_k with $k\geq 1$ (with some additional care necessary when $k=1$, see the end of the proof). By examining the explicit expression (2.5) we see that the product of kernels K_λ has the form

$$K_\lambda(x,y_0)\dots K_\lambda(y_{k-1},y_k) = \sum\frac{1}{(2i)^{k+1}}e^{i\lambda A}e^{i\rho_+B}\sigma(\lambda) \tag{3.64}$$

where:

1. the number of terms in the sum does not exceed 2^{k+1};
2. the quantities A, B are linear combinations of $x, y_0, \dots, y_k$ with the property that

$$A\geq 0, \qquad B\geq 0;$$

(3) the functions $\sigma(\lambda)$ are products of the form

$$\sigma(\lambda) = \frac{1}{\lambda^\ell}\left(\frac{1}{\lambda}\frac{\lambda-\rho_+}{\lambda+\rho_+}\right)^m \frac{1}{(\lambda+\rho_+)^n}\frac{1}{\rho_+^p}\left(\frac{1}{\rho_+}\frac{\lambda-\rho_+}{\lambda+\rho_+}\right)^q$$

where the non negative integers ℓ, m, n, p, q satisfy

$$\ell + m + n + p + q = k + 1.$$

We elaborate a little on the properties of the functions σ. Notice that we are in the region $|\lambda| \geq \lambda_0 > 1$. Using the identity $\lambda - \rho_+ = (\lambda + \rho_+)^{-1}$ we rewrite the expression of σ in the simpler form

$$\sigma(\lambda) = \frac{1}{\lambda^{\ell+m}}\frac{1}{\rho_+^{p+q}}\frac{1}{(\lambda+\rho_+)^{n+2m+2q}}.$$

In particular, we notice that $\sigma(\lambda)$ is monotone decreasing on $\lambda > 1$ and monotone increasing on $\lambda < -1$, and for $|\lambda| \geq \lambda_0$ we have (since $|\lambda| \geq |\rho_+|$ and $\rho_+(\lambda_0) > 1$)

$$|\sigma(\lambda)| \leq \rho_+(\lambda_0)^{-(\ell+p+n+3m+3q)} \leq \rho_+(\lambda_0)^{-k-1}. \tag{3.65}$$

Thus $\sigma(\lambda)$ is monotone and satisfies the bound (3.65); when $k \geq 2$, a direct application of Lemma 3.3, keeping into account that $A, B \geq 0$ and also the additional factor λ (which is bounded by $2\rho_+(\lambda)$ with our choice of λ_0) gives

$$\left|\int e^{it\lambda^2}e^{i\lambda A}e^{i\rho_+(\lambda)B}\sigma(\lambda)\lambda\chi(\lambda)d\lambda\right| \leq 240\rho_+(\lambda_0)^{-k}.$$

Summing the estimates over all the terms in (3.64) and noticing the power of 2 at the denominator, we conclude the proof of (3.62) with a constant $C = 240$.

In the case $k = 1$ there is an additional technical difficulty due to the fact that the convergence of the integral in λ must be justified since the integrand decays like λ^{-1} only. To this end it is sufficient to approximate $A_1 f$ by introducing an additional cutoff of the form $\chi(\lambda/M)$ and noticing that the estimate is uniform as $M \to \infty$.

The proof of the high energy case is concluded.

Appendix A. Two lemmas

Lemma A.1. *For all $\chi \in C_c^1(\mathbb{R})$ and $A \neq 0$,*

$$\left|\int_0^\infty e^{i\lambda A}\chi(\lambda)\lambda^{-1/2}d\lambda\right| \leq 5\|\chi'\|_{L^1}\cdot|A|^{-1/2}. \tag{A.1}$$

Proof. Notice the inequality

$$\|\chi\|_{L^\infty} \leq \|\chi'\|_{L^1}$$

since χ is compactly supported. We split the integral as $\int_0^\epsilon + \int_\epsilon^{+\infty}$ and we calculate

$$\left|\int_0^\epsilon\right| \leq \|\chi\|_{L^\infty}\int_0^\epsilon \lambda^{-1/2}d\lambda \leq 2\epsilon^{1/2}\|\chi'\|_{L^1}.$$

For the remaining piece, integration by parts gives

$$iA\int_\epsilon^\infty = -e^{i\epsilon A}\chi(\epsilon)\epsilon^{-1/2} - \int_\epsilon^\infty e^{i\lambda A}\partial_\lambda(\chi\lambda^{-1/2})$$

so that

$$\left|A\int_\epsilon^\infty\right| \leq \|\chi\|_{L^\infty}\epsilon^{-1/2} + \int_\epsilon^\infty |\chi'|\lambda^{-1/2} + \frac{1}{2}\int_\epsilon^\infty |\chi|\lambda^{-3/2} \leq 3\|\chi'\|_{L^1}\epsilon^{-1/2}.$$

Thus the complete integral satisfies

$$\left|\int_0^\infty\right| \leq \|\chi'\|_{L^1}\left(2\epsilon^{1/2} + \frac{3}{\epsilon^{1/2}A}\right)$$

and chosing $\epsilon = 3/(2A)$ we obtain (A.1). $\square$

Lemma A.2. *For all* $\chi \in C_c^2(\mathbb{R})$ *and* $A \neq 0$,

$$\left| \int_0^\infty e^{i\lambda A} \chi(\lambda) \lambda^{1/2} d\lambda \right| \leq 5 \|\chi''/2 + \lambda \chi'\|_{L^1} \cdot |A|^{-3/2}. \tag{A.2}$$

Proof. It is sufficient to apply the previous Lemma to the identity

$$iA \int_0^\infty = - \int_0^\infty e^{i\lambda A} \left(\lambda \chi' + \chi/2 \right) \lambda^{-1/2} d\lambda.$$

$\square$

References

[1] Tuncay Aktosun. On the Schrödinger equation with steplike potentials. *J. Math. Phys.*, 40(11):5289–5305, 1999.

[2] Galtbayar Artbazar and Kenji Yajima. The L^p-continuity of wave operators for one dimensional Schrödinger operators. *J. Math. Sci. Univ. Tokyo*, 7(2):221–240, 2000.

[3] Anne Boutet de Monvel, Iryna Egorova, and Gerald Teschl. Inverse scattering theory for one-dimensional Schrödinger operators with steplike finite-gap potentials. *J. Anal. Math.*, 106:271–316, 2008.

[4] Vladimir Buslaev and Vladimir Fomin. An inverse scattering problem for the one-dimensional Schrödinger equation on the entire axis. *Vestnik Leningrad. Univ.*, 17(1):56–64, 1962.

[5] Michael Christ and Alexander Kiselev. WKB and spectral analysis of one-dimensional Schrödinger operators with slowly varying potentials. *Comm. Math. Phys.*, 218(2):245–262, 2001.

[6] Michael Christ and Alexander Kiselev. Scattering and wave operators for one-dimensional Schrödinger operators with slowly decaying nonsmooth potentials. *Geom. Funct. Anal.*, 12(6):1174–1234, 2002.

[7] Tanya Christiansen. Resonances for steplike potentials: forward and inverse results. *Trans. Amer. Math. Soc.*, 358(5):2071–2089 (electronic), 2006.

[8] Amy Cohen. Solutions of the Korteweg-de Vries equation with steplike initial profile. *Comm. Partial Differential Equations*, 9(8):751–806, 1984.

[9] Amy Cohen and Thomas Kappeler. Scattering and inverse scattering for steplike potentials in the Schrödinger equation. *Indiana Univ. Math. J.*, 34(1):127–180, 1985.

[10] Piero D'Ancona and Luca Fanelli. L^p-boundedness of the wave operator for the one dimensional Schrödinger operator. *Comm. Math. Phys.*, 268(2):415–438, 2006.

[11] Piero D'Ancona, Vittoria Pierfelice, and Alessandro Teta. Dispersive estimate for the Schrödinger equation with point interactions. *Math. Methods Appl. Sci.*, 29(3):309–323, 2006.

[12] Edward Brian Davies and Barry Simon. Scattering theory for systems with different spatial asymptotics on the left and right. *Comm. Math. Phys.*, 63(3):277–301, 1978.

[13] Percy Deift and Eugene Trubowitz. Inverse scattering on the line. *Comm. Pure Appl. Math.*, 32(2):121–251, 1979.

[14] Ludwig Dmitrievich Faddeyev. The inverse problem in the quantum theory of scattering. *J. Mathematical Phys.*, 4:72–104, 1963.

[15] Fritz Gesztesy. Scattering theory for one-dimensional systems with nontrivial spatial asymptotics. In *Schrödinger operators, Aarhus 1985*, volume 1218 of *Lecture Notes in Math.*, pages 93–122. Springer, Berlin, 1986.

[16] Fritz Gesztesy, Raymond Nowell, and Walter Pötz. One-dimensional scattering theory for quantum systems with nontrivial spatial asymptotics. *Differential Integral Equations*, 10(3):521–546, 1997.

[17] Michael Goldberg and Wilhelm Schlag. Dispersive estimates for Schrödinger operators in dimensions one and three. *Comm. Math. Phys.*, 251(1):157–178, 2004.

[18] Jean-Lin Journé, Avraham Soffer, and Christopher D. Sogge. Decay estimates for Schrödinger operators. *Comm. Pure Appl. Math.*, 44(5):573–604, 1991.

[19] Thomas Kappeler. Solutions of the Korteweg-de Vries equation with steplike initial data. *J. Differential Equations*, 63(3):306–331, 1986.

[20] Elias M. Stein. Oscillatory integrals in Fourier analysis. In *Beijing lectures in harmonic analysis (Beijing, 1984)*, volume 112 of *Ann. of Math. Stud.*, pages 307–355. Princeton Univ. Press, Princeton, NJ, 1986.

[21] Gerald Teschl. *Mathematical methods in quantum mechanics*, volume 99 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2009. With applications to Schrödinger operators.

[22] Ricardo Weder. L^p-$L^{p'}$ estimates for the Schrödinger equation and inverse scattering. In *Differential equations and mathematical physics (Birmingham, AL, 1999)*, volume 16 of *AMS/IP Stud. Adv. Math.*, pages 435–448. Amer. Math. Soc., Providence, RI, 2000.

[23] Joachim Weidmann. *Linear operators in Hilbert spaces*, volume 68 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1980. Translated from the German by Joseph Szücs.

[24] Joachim Weidmann. *Spectral theory of ordinary differential operators*, volume 1258 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 1987.

[25] Kenji Yajima. The $W^{k,p}$-continuity of wave operators for Schrödinger operators. *J. Math. Soc. Japan*, 47(3):551–581, 1995.

[26] Kenji Yajima. The $W^{k,p}$-continuity of wave operators for schrödinger operators. iii. even-dimensional cases $m \geq 4$. *J. Math. Sci. Univ. Tokyo*, 2(2):311–346, 1995.

[27] Kenji Yajima. L^p-boundedness of wave operators for two-dimensional Schrödinger operators. *Comm. Math. Phys.*, 208(1):125–152, 1999.

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